Cancellation in entropic algebras

M. M. Stronkowski

Abstract. We describe the equational theory of the class of cancellative entropic algebras of a fixed type. We prove that a cancellative entropic algebra embeds into an entropic polyquasigroup, a natural generalization of a quasigroup. In fact our results are even more general and some corollaries hold also for non-entropic algebras. For instance an algebra with a binary cancellative term operation, which is a homomorphism, is quasi-affine. This gives a strengthening of K. Kearnes' theorem. Our results generalize theorems obtained earlier by M. Sholander and J. Ježek, T. Kepka in the case of groupoids.

1. Introduction

An algebra (A, Ω) is *entropic* if each basic n-ary operation $\omega \in \Omega$ is a homomorphism from (A^n, Ω) into (A, Ω). In the groupoid case this reduces to the satisfaction of the identity

$$(xy)(zt) \approx (xz)(yt).$$

Interesting results for such groupoids were obtained by J. Ježek and T. Kepka. In particular they described the equational theory of the class of cancellative entropic groupoids [8, 9]. Based on this, they provided a new proof of Sholander's theorem [24], stating that each entropic cancellative groupoid embeds into an entropic quasigroup. In this paper we follow the path further and carry over both results to the case of algebras of any type.

The main point of our interest is to generalize the following result of A. Romanowska and J. Smith.

Theorem 1.1 ([22, 23])**.** *Each cancellative mode (idempotent entropic algebra) is a subreduct of a module over a commutative ring.*

Our present aim is to remove idempotency from the assumption. Results obtained in this paper are used for this purpose in [27, 28].

At the beginning we introduce strongly entropic algebras which form (except for the trivial case of algebras with only unary and nullary operations) a proper subvariety of the variety of entropic algebras. We introduce the concept of a monoid of

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terms, and for such a monoid $\mathcal{M}$, the property of $\mathcal{M}$-cancellativity. By choosing the appropriate monoid of terms, as particular cases we obtain cancellativity, left (right) cancellativity (for groupoids), and many other "cancellation-like" properties. We introduce $\mathcal{M}$-polyquasigroups, a natural generalization of quasigroups. In Theorem 9.3 we prove that for a proper (see Section 5) monoid $\mathcal{M}$ of terms, the class of $\mathcal{M}$-cancellative entropic algebras generates the variety of strongly entropic algebras. This is one of our main results. For each monoid $\mathcal{M}$ of terms free strongly entropic $\mathcal{M}$-cancellative algebras, which are in fact free strongly entropic algebras, have a natural representation as subreducts of vector spaces. Using this representation, it is shown that these algebras embed into strongly entropic $\mathcal{M}$-polyquasigroups. Let (F, Ω) be a free $\mathcal{M}$-cancellative strongly entropic algebra, and let $(F, \Omega) \hookrightarrow (G, \Omega)$ be an embedding into a strongly entropic $\mathcal{M}$-polyquasigroup. Let $(F/\theta, \Omega)$, where θ is a congruence of (F, Ω), be cancellative. We prove that θ may be extended to a congruence $\tilde{\theta}$ of (G, Ω) such that $(G/\tilde{\theta}, \Omega)$ is cancellative as well. Thus again $(F/\theta, \Omega) \hookrightarrow (G/\tilde{\theta}, \Omega)$ is an embedding into a strongly entropic $\mathcal{M}$-polyquasigroup. This embedding preserves the subdirect irreducibility and the satisfaction of identities. Additionally, in the idempotent (mode) case, it preserves the simplicity of algebras and the satisfaction of universal sentences, such as quasi-identities (Theorem 7.1). Combining Theorem 9.3 and Theorem 7.1, we see that for each proper monoid $\mathcal{M}$ of terms, each $\mathcal{M}$-cancellative entropic algebra embeds into an entropic $\mathcal{M}$-polyquasigroup. Sholander's theorem may be recognized as a special case of this. The localization of modules over integral domains, and embedability of cancellative commutative unars into commutative bijective unars [10], are contained in our theorem as well. By the functoriality of the construction described above, we may extend embedability results even to some non-entropic algebras (Theorem 9.5). As a corollary, we obtain quasi-affine representations for some algebras (Theorem 8.6).

We warn the reader that the terminology in the field is not fixed. For instance entropic groupoids are called medial in [1, 8, 9, 11, 13, 19], abelian in [2, 16], and alternation in [24]. Strongly entropic groupoids were considered in [8, 9, 19] under the name of entropic groupoids.

2. Basic definitions and notation

By $\mathbb{N}$ we denote the set of natural numbers. We fix a similarity type $\tau\colon \Omega \to \mathbb{N}$. When we consider algebras or terms not referring to their types we mean that their types coincide with τ. In particular if we write "the variety of entropic algebras" we actually mean "the variety of entropic algebras of the type τ". It will be convenient to fix an infinite countable set X of variables. We assume that X is disjoint with Ω.

An algebra (A, Ω) is *entropic* if each basic operation $\omega \in \Omega$ is a homomorphism from $(A^{\tau(\omega)}, \Omega)$ into (A, Ω). This is equivalent to the satisfaction of all *entropic* *identities*

$$\mu(\nu(x_1^1,\dots,x_{\tau(\nu)}^1),\dots,\nu(x_1^{\tau(\mu)},\dots,x_{\tau(\nu)}^{\tau(\mu)})) \approx \nu(\mu(x_1^1,\dots,x_1^{\tau(\mu)}),\dots,\mu(x_{\tau(\nu)}^1,\dots,x_{\tau(\nu)}^{\tau(\mu)})), \quad (\varepsilon_{\mu,\nu})$$

for $\mu,\nu \in \Omega$. By induction an identity

$$t(s(x_1^1,\dots,x_n^1),\dots,s(x_1^m,\dots,x_n^m)) \approx s(t(x_1^1,\dots,x_1^m),\dots,t(x_n^1,\dots,x_n^m)) \quad (\varepsilon_{t,s})$$

must hold in entropic algebras for each terms t and s. Moreover, each constant considered as an element selected by a nullary operation forms a subalgebra of (A,Ω). Hence all constants coincide. Thus, without loss of generality, we may assume that Ω has at most one symbol of a nullary operation. It will be denoted by o. The variety of entropic algebras (of the type τ) is denoted by $\boldsymbol{E}$.

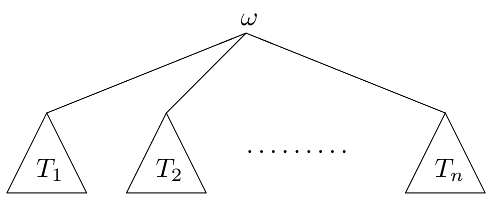
Among entropic algebras, modes deserve special attention. A thorough investigation of them was undertaken in [21, 23] An algebra of a plural type is a *mode* if it is entropic and *idempotent*, that is all *idempotent identities*

$$\omega(x,\dots,x) \approx x, \quad (\iota_\omega)$$

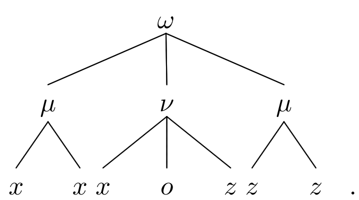
where $\omega \in \Omega$, hold in it. While entropic algebras are characterized by the fact that all their term operations are homomorphisms, modes are characterized by the property that all their polynomial operations are homomorphisms.

The absolutely free algebra (of type τ) generated by a set X is denoted by $(\mathrm{Term}(X),\Omega)$. Its elements are represented as terms with variables in X. A term t is *linear relative to a variable* x if x occurs in t exactly once. A term is *linear* if it is linear relative to all variables occurring in it. The set of all variables occurring in a term t is denoted by $\arg(t)$.

It is convenient to think about terms as rooted trees. The correspondence is given inductively as follows. Term y which is a variable or a constant is represented by the tree with only one node, a root, labelled by y. If $t = \omega(t_1,\dots,t_n)$, where $\tau(\omega) = n \geqslant 1$ and $t_1,\dots,t_n$ are terms represented by trees $T_1,\dots,T_n$ respectively, then the term t is represented by the tree



For instance the term $\omega(\mu(x,x),\nu(x,o,z),\mu(z,z))$ is represented by the tree



The *depth* of a term is the length of the longest branch in the tree representing it.

By an *identity* we mean a pair of terms. Generally, we denote an identity (t, s) by $t \approx s$. An identity is *linear* if the terms on both sides of it are linear. Equational theories are identified with fully invariant congruences of $(\mathrm{Term}(X), \Omega)$. An equational theory is *linear* if it has an equational basis consisting of linear identities. Sometimes, if we want to emphasize that the identity $t \approx s$ belongs to some relation (equational theory) $R \subseteq \mathrm{Term}(X)^2$ we write $t\, R\, s$. For instance instead of $\boldsymbol{E} \models t \approx s$ we rather write $t \overset{e}{\approx} s$.

Let us distinguish some subsets of Ω. For a natural number i let

$$\Omega_i = \{\omega \in \Omega \mid \tau(\omega) = i\} = \tau^{-1}(i),$$
$$\Omega_{>i} = \{\omega \in \Omega \mid \tau(\omega) > i\} = \tau^{-1}(\{j \in \mathbb{N} \mid i < j\}).$$

In particular, Ω_0 is the set of all nullary operation symbols and $\Omega_{>0}$ is the set of all non-nullary operation symbols. For $\Delta \in \{i, > j \mid i, j \in \mathbb{N}\}$ let

$$\tau_\Delta = \tau|_{\Omega_\Delta}.$$

Algebras (A, Ω) and (A, Φ) are (polynomially) equivalent if they have the same (polynomial) term operations. An algebra (A, Ω) is a reduct of (A, Φ) if each basic operation of (A, Ω) is a term operation of (A, Φ). A subreduct is a subalgebra of a reduct.

3. Strongly entropic algebras

Here we introduce strongly entropic algebras, a crucial class in our considerations, and present basic properties of the variety formed by them. But first we need more definitions.

Roughly speaking, semirings are rings without subtraction. Precisely, a *semiring* is an algebra $(R, +, 0, \cdot, 1)$ such that $(R, +, 0)$ is a commutative monoid, $(R, \cdot, 1)$ is a monoid, $0x = x0 = 0$ and multiplication distributes over addition. A semiring is *commutative* if the multiplication is commutative. The non-commutative semiring $(\mathbb{N}\langle V\rangle, +, 0, \cdot, 1)$ of polynomials with non-commuting indeterminants in V and natural coefficients is a free semiring over V. One may construct it as follows: let $(V^*, \cdot, 1)$ be a free monoid over V, next let $(\mathbb{N}\langle V\rangle, +, 0)$ be a free commutative monoid over V^* and finally extend multiplication using $0x = x0 = 0$ and distributivity. Similarly, we represent a free commutative semiring over V as the commutative semiring of polynomials with commuting indeterminants in V and natural coefficient. It is denoted by $(\mathbb{N}[V], +, 0, \cdot, 1)$. Let

$$\bar{\ } \colon (\mathbb{N}\langle V\rangle, +, 0, \cdot, 1) \to (\mathbb{N}[V], +, 0, \cdot, 1)$$

be a semiring homomorphism which is the identity mapping on V.

By a *semimodule* over a semiring $(R, +, 0, \cdot, 1)$ we mean an algebra $(M, +, 0, R)$, where the unary operations determined by elements of R are endomorphisms of the commutative monoid $(M, +, 0)$ and moreover

$$1m = m,$$
$$0m = 0,$$
$$(r_1 \cdot r_2)m = r_1(r_2 m),$$
$$(r_1 + r_2)m = r_1 m + r_2 m.$$

For further information on semirings and semimodules, we refer the reader to [6].

The *address* $\mathrm{a}(t, y)$ of $y \in X \cup \Omega_0$ in a term t says how y is placed in t. The precise definition is as follows. Put

$$\Sigma = \big\{(\omega, i) \mid \omega \in \Omega_{>0} \ \text{ and } \ 1 \leqslant i \leqslant \tau(\omega)\big\},$$

and let

$$\mathrm{a}\colon \operatorname{Term}(X) \times (X \cup \Omega_0) \to \mathbb{N}\langle\Sigma\rangle$$

be a function given inductively by

$$\mathrm{a}(x, y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

for $x, y \in X \cup \Omega_0$, and further by

$$\mathrm{a}(\omega(t_1, \dots, t_{\tau(\omega)}), y) = \sum_{i=1}^{\tau(\omega)} (\omega, i)\, \mathrm{a}(t_i, y)$$

for $\omega \in \Omega_{>0}$. For example consider a term

$$t = \mu(\nu(x, o), \mu(o, x)).$$

Then $\mathrm{a}(t, o) = (\mu, 1)(\nu, 2) + (\mu, 2)(\mu, 1)$ and $\mathrm{a}(t, x) = (\mu, 1)(\nu, 1) + (\mu, 2)(\mu, 2)$. Note that if $\mathrm{a}(t, y) = \mathrm{a}(s, y)$ for all $y \in X \cup \Omega_0$, then $s = t$. Moreover for a term t and some distinct variables x and y their addresses in t coincide if and only if they are both equal to 0, that is x and y do not occur in t. Indeed, x occurs in t iff $\mathrm{a}(t, x) \neq 0$.

Finally, we are ready to give the main definition of this section. Recall that $\bar{\ }\colon \mathbb{N}\langle\Sigma\rangle \to \mathbb{N}[\Sigma]$; $u \mapsto \bar{u}$ is the semiring homomorphism which maps each (ω, i) in Σ to itself. Put $\bar{\mathrm{a}} = \bar{\ } \circ \mathrm{a}$. For terms $t, s \in \operatorname{Term}(X)$ let

$$t \overset{se}{\approx} s \quad \text{iff} \quad \forall x \in X \ \ \bar{\mathrm{a}}(t, x) = \bar{\mathrm{a}}(s, x).$$

An algebra is *strongly entropic* if it satisfies all identities from $\overset{se}{\approx}$. The variety of strongly entropic algebras is denoted by $\boldsymbol{SE}$.

We start with the construction of free strongly entropic algebras. For a set Y we define $(N(Y), +, 0, \mathbb{N}[\Sigma])$ to be the semimodule over the semiring $(\mathbb{N}[\Sigma], +, 0, \cdot, 1)$ freely generated by the set Y. We equip the set $N(Y)$ with a τ-structure by putting

$$\omega(p_1, \dots, p_{\tau(\omega)}) = (\omega, 1)p_1 + \ldots + (\omega, \tau(\omega))p_{\tau(\omega)}$$

for $\omega \in \Omega_{>0}$ and

$$o = 0$$

for $o \in \Omega_0$. Denote by $(P(Y), \Omega)$ the subalgebra of $(N(Y), \Omega)$ generated by Y.

Proposition 3.1. *The relation* $\stackrel{se}{\approx}$ *equals to the equational theory of* $(N(X),\Omega)$ *and* $(P(X),\Omega)$*. Moreover, for a set* Y *the algebra* $(P(Y),\Omega)$ *is free in* $\boldsymbol{SE}$ *over* Y*.*

Proof. For a term $t(x_1,\dots,x_n)$ and elements $p_1,\dots,p_n \in N(Y)$ we have

$$t(p_1,\dots,p_n) = \sum_{i=1}^{n} \bar{\mathrm{a}}(t,x_i)p_i.$$

Thus, $(N(Y),\Omega)$ and $(P(Y),\Omega)$ are strongly entropic. Now consider an element $p = \sum_{i=1}^{n} r_i y_i$ of $N(Y)$, where $r_i \in \mathbb{N}[\Sigma]$ and $y_i \in Y$. Note that $p \in P(Y)$ iff there is a term $t(x_1,\dots,x_n)$ such that $\mathrm{a}(t,x_i) = r_i$. If there is another term $s(x_1,\dots,x_n)$ such that $\mathrm{a}(s,x_i) = r_i$ then $t \stackrel{se}{\approx} s$. Let us consider a mapping $f\colon Y \to A$, where (A,Ω) is a strongly entropic algebra. Put

$$\tilde{f}\colon P(Y) \to A; \quad \sum_{i=1}^{n} \bar{\mathrm{a}}(t,x_i)y_i \mapsto t(f(y_1),\dots,f(y_n)).$$

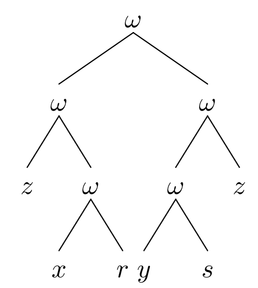
By the remark above $\tilde{f}$ well defined. It is straightforward to check that it is also a homomorphism extending f. Summarizing $(P(Y),\Omega)$ is free in $\boldsymbol{SE}$ over Y. Putting $Y = X$ we obtain first statement of Proposition. □

Proposition 3.2. *The variety* $\boldsymbol{SE}$ *has a linear equational theory.*

Proof. Consider an identity $t \stackrel{se}{\approx} s$. Let $\mathrm{a}(t,x) = \sum_{j=1}^{k_x} \alpha_j^x$ and $\mathrm{a}(s,x) = \sum_{j=1}^{k_x} \beta_j^x$. Because $\bar{\mathrm{a}}(t,x) = \bar{\mathrm{a}}(s,x)$, we may assume that $\overline{\alpha_j^x} = \overline{\beta_j^x}$ for all $1 \leqslant j \leqslant k_x$. Now consider a family of mutually distinct variables y_j^x, where $x \in X$ and $1 \leqslant j \leqslant k_x$. Let t' and s' be terms such that $\mathrm{a}(t',y_j^x) = \alpha_j^x$, $\mathrm{a}(t',o) = \mathrm{a}(t,o)$ and $\mathrm{a}(s',y_j^x) = \beta_j^x$ and $\mathrm{a}(s',o) = \mathrm{a}(s,o)$. Obviously, t' and s' are uniquely determined linear terms. Moreover $t' \stackrel{se}{\approx} s'$. Finally, note that the identity $t \approx s$ is a consequence of the identity $t' \approx s'$. □

Proposition 3.3. *We have the inequality* $\boldsymbol{SE} \leqslant \boldsymbol{E}$*. The equality* $\boldsymbol{SE} = \boldsymbol{E}$ *holds iff* $\Omega_{>1} = \emptyset$*.*

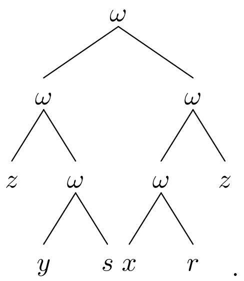
Proof. The only nontrivial implication is that if $\Omega_{>1} \neq \emptyset$, then $\boldsymbol{E} \not\leqslant \boldsymbol{SE}$. So assume that there is n-ary operation symbol $\omega \in \Omega_{>1}$. We start with the case $n = 2$. Let $t = t(x,y,z,r,s)$ be a term represented by the tree



and consider the identity

$$t(x,y,z,r,s) \overset{se}{\approx} t(y,x,z,r,s).$$

We cannot apply entropic law to any proper subterm of t. Thus, if an identity $t \approx s$ follows from entropicity then s coincides with t or is represented by the tree



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In both cases $s \neq t(y,x,z,r,s)$. For $n > 2$ we proceed similarly, but in this case it is enough to consider a simpler identity

$$\omega(\omega(z,x,z,\dots,z),\omega(y,z,\dots,z),z\dots,z) \overset{se}{\approx} \omega(\omega(y,x,z,\dots,z),\omega(x,z,\dots,z),z\dots,z).$$

□

In [9, Example 2.4.1.] a six-element entropic groupoid, which is not strongly entropic, is presented.

Note that each subreduct of a semimodule over a commutative semiring is strongly entropic. On the other, hand if an algebra (N,Ω) is a reduct of an algebra which is polynomially equivalent to a semimodule $(N,+,0,R)$ over a commutative semiring $(R,+,0,\cdot,1)$ then (N,Ω) does not have to be even entropic in general. But if $|\Omega| = 1$ and $\Omega_0 = \emptyset$ then it is again strongly entropic.

Proposition 3.4. *Let $(N,+,0,R)$ be a semimodule over a commutative semiring $(R,+,0,\cdot,1)$. An algebra (A,ω), where $A \subseteq N$ and*

$$\omega(a_1,\dots,a_n) = r_1a_1 + \ldots + r_na_n + c \tag{3.1}$$

for some $r_1,\dots,r_n \in R$ and $c \in N$, is strongly entropic.

Proof. For the purpose of this proof we assume that $\Omega = \{\omega\}$. Denote by $\mathcal{I}$ the equational theory with the basis consisting of all identities of the form

$$t(x,y,z_1,\dots,z_n) \approx t(y,x,z_1,\dots,z_n),$$

where t is a linear term and $\bar{\mathrm{a}}(t,x) = \bar{\mathrm{a}}(t,y)$. First, we will show that $\mathcal{I}$ coincides with $\overset{se}{\approx}$. This part of the proof is based on the idea from [9, Section 2.1.]. Let $t(x_1,\dots,x_n) \overset{se}{\approx} s(x_1,\dots,x_n)$ be a linear identity. We may assume that if the length of $\mathrm{a}(t,x_i)$ is smaller than the length of $\mathrm{a}(t,x_j)$ then $i < j$. We will construct inductively a sequence of terms $t_0,\dots,t_n$ such that $t_0 = t$, $t_n = s$, for all $i < n$ the identity $t_i \approx t_{i+1}$ belongs to $\mathcal{I}$ and $\mathrm{a}(t_k,x_j) = \mathrm{a}(s,x_j)$ for all $j \leqslant k \leqslant n$. If such a sequence exists then obviously the identity $t \approx s$ must belong to $\mathcal{I}$. The first step is easy: $t_0 = t$. Assume that $t_0, \dots, t_{k-1}$ are already defined. Let $\mathrm{a}(t_{k-1}, x_k) = \alpha$ and $\mathrm{a}(s, x_k) = \beta$. If $\alpha = \beta$ then we put $t_k = t_{k+1}$. For $\alpha \neq \beta$ observe that for $j < k$ the address $\mathrm{a}(t_{k-1}, x_j) = \mathrm{a}(s, x_j)$ cannot be a prefix of β. On the other hand for all $l > k$ the length of $\mathrm{a}(t_{k-1}, x_l)$ is at least as big as the length of $\mathrm{a}(t_{k-1}, x_k)$. This and the fact that $|\Omega| = 1$ yield the existence of a linear term $t'(x_1, \dots, x_k, y, z_1, \dots, z_n)$ such that $\mathrm{a}(t', y) = \beta$ and $t_{k-1} = t'(x_1, \dots, x_k, w, z_1, \dots, z_n)$ for some term w. Define t_k to be a term $t'(x_1, \dots, x_{k-1}, w, x_k, z_1, \dots, z_n)$. Thus, we have proved that the linear part of $\overset{se}{\approx}$ is contained in $\mathcal{I}$. By Proposition 3.2, this yields that $\overset{se}{\approx} = \mathcal{I}$.

Now we go to the second part of the proof. Let

$$\rho\colon (\mathbb{N}[\Sigma], +, 0, \cdot, 1) \to (R, +, 0, \cdot, 1)$$

be a semiring homomorphism which sends each (ω, i) to r_i. Extending the definition (3.1) to a term operation we obtain for $t(x_1, \dots, x_n)$ and elements $a_1, \dots, a_n \in A$

$$t(a_1, \dots, a_n) = \sum_{i=1}^{n} \rho \circ \bar{\mathrm{a}}(t, x_i) a_i + c_t,$$

where $c_t = t(0, \dots, 0)$ is an element of N, which does not depend on the elements $a_1, \dots, a_n$. So if $s(x_1, x_2, \dots, x_n) = t(x_2, x_1, \dots, x_n)$ and $\bar{\mathrm{a}}(t, x_1) = \bar{\mathrm{a}}(t, x_2)$ then $t(a_1, \dots, a_n) = s(a_1, \dots, a_n)$. Thus, (N, ω) satisfies all identities from $\mathcal{I}$ and hence it is strongly entropic. □

In the and of this section, we note that the variety of strongly entropic algebras is not finitely based (except in trivial cases). In [19] an independent basis was obtained for the case of algebras with one n-ary basic operation.

4. Manipulation of terms

Let

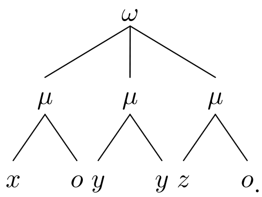
$$\mathrm{b}\colon \mathrm{Term}(X) \times (X \cup \Omega_0) \to \mathbb{N}\langle \Omega_{>0} \rangle$$

and

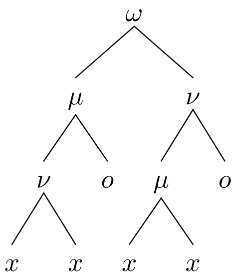
$$\bar{\mathrm{b}}\colon \mathrm{Term}(X) \times (X \cup \Omega_0) \to \mathbb{N}[\Omega_{>0}]$$

be mappings given analogously as a and ā respectively. Simply replace each (ω, i) by ω in the definitions.

A term t is *isosceles* if there is a word $\gamma \in \Omega_{>0}^*$ such that for each $y \in X \cup \Omega_0$, $\mathrm{b}(t, y) = k\gamma$ for some $k \in \mathbb{N}$. The word γ is called the *trace* of t. This condition says that the term t has a very regular form. All variables and all constants are on the same lowest level. On each level, except the lowest one, there is exactly one operation symbol from $\Omega_{>0}$. Notice that if $|\Omega_{>0}| = 1$ then a given term is isosceles if and only if it is full. As an example consider a term represented by the tree



It is isosceles while the one represented by the tree



(4.1)

is not. Still, as we will see, it is easy to repair it.

Recall that $t \overset{e}{\approx} s$ means that the identity $t \approx s$ holds in all entropic algebras.

Lemma 4.1. *Let $\omega \in \Omega_{>0}$ and t be a term. Assume that for each $x \in X$ there exists $u \in \mathbb{N}[\Omega_{>0}]$ with $\bar{\mathrm{b}}(t, x) = \omega u$. Then there are terms $t_1, \ldots, t_{\tau(\omega)}$ such that $t \overset{e}{\approx} \omega(t_1, \ldots, t_{\tau(\omega)})$.*

Proof. By the assumption, there are terms s and s_j^i such that

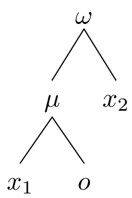
$$t = s(\omega(s_1^1, \ldots, s_{\tau(\omega)}^1), \ldots, \omega(s_1^n, \ldots, s_{\tau(\omega)}^n))$$

and thus by entropicity

$$t \overset{e}{\approx} \omega(s(s_1^1, \ldots, s_1^n), \ldots, s(s_{\tau(\omega)}^1, \ldots, s_{\tau(\omega)}^n)) = \omega(t_1, \ldots, t_{\tau(\omega)}),$$

where $t_i = s(s_i^1, \ldots, s_i^n)$. □

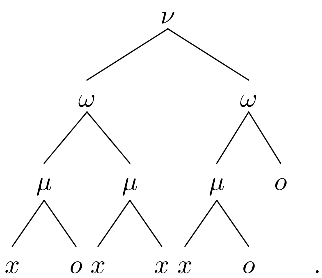
To illustrate how Lemma 4.1 works consider again the term t represented by the tree (4.1). We would like to show that $t \overset{e}{\approx} \nu(t_1, t_2)$ for some terms t_1, t_2. Let $s(x_1, x_2)$ be a term given by the tree



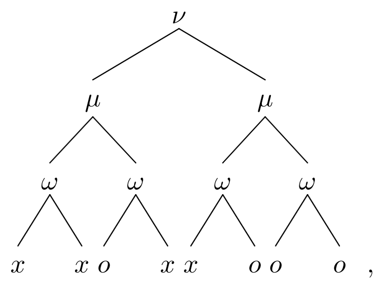
Then

$$t = s(\nu(x, x), \nu(\mu(x, x), o)) \overset{e}{\approx} \nu(s(x, \mu(x, x)), s(x, o)) = \nu(t_1, t_2).$$

The latter term in this identity is represented by the tree



Now we apply the "pulling up technology" to the terms t_1 and t_2 in order to get μ on the top of them. We obtain the term t' given by the tree



which is isosceles with the trace $\nu\mu\omega$ and moreover $t \overset{e}{\approx} t'$. This illustrates our next key lemma. Recall that

$$\bar{\ }: (\mathbb{N}\langle\Omega_{>0}\rangle, +, 0, \cdot, 1) \to (\mathbb{N}_C[\Omega_{>0}], +, 0, \cdot, 1); \quad r \to \bar{r}$$

is a semiring homomorphism such that $\bar{\omega} = \omega$ for all $\omega \in \Omega_{>0}$.

Lemma 4.2. *Let $\gamma \in \Omega^*_{\geqslant 0}$ and let t be a term such that for all $x \in X$ there is a natural number k with $\bar{\mathrm{b}}(t, x) = k\bar{\gamma}$. Then there is an isosceles term t' with the trace γ such that $t \overset{e}{\approx} t'$.*

Proof. We proceed by induction on the length of γ. If it is equal to 0 then t is a variable and we may put $t' = t$ or t is a constant term and hence $t \overset{e}{\approx} o \overset{e}{\approx} t'$, where t' is the constant isosceles term with the trace γ. So assume that $\gamma = \omega\gamma'$. Then the assumption of Lemma 4.1 is satisfied, so that $t = \omega(t_1, \dots, t_{\tau(\omega)})$ for some terms $t_1, \dots, t_{\tau(\omega)}$. Note however, that for each $1 \leqslant i \leqslant \tau(\omega)$ and $x \in X$ there is a natural number l such that $\bar{\mathrm{b}}(t_i, x) = l\bar{\gamma'}$. Thus, by the induction assumption, there are isosceles terms t'_i with the trace γ' such that $t_i \overset{e}{\approx} t'_i$ for $1 \leqslant i \leqslant \tau(\omega)$, whence

$$t \overset{e}{\approx} \omega(t_1, \dots, t_{\tau(\omega)}) \overset{e}{\approx} \omega(t'_1, \dots, t'_{\tau(\omega)}) = t'.$$

Obviously, t' is an isosceles term with the trace γ. □

Corollary 4.3. *Let t be a linear term such that $\bar{\mathrm{b}}(t, x) = \bar{\mathrm{b}}(t, y)$ for all $x, y \in \arg(t)$. Then there is an isosceles linear term t' such that $t \overset{e}{\approx} t'$.*

We will frequently use Lemmas 4.1, 4.2 and Corollary 4.3 without referring to them.

Proposition 4.4. *Let $t_1(x_1,\dots,x_n)$ and $t_2(x_1,\dots,x_n)$ be linear terms and assume that $t_1 \overset{se}{\approx} t_2$. Then there exist linear isosceles $\tau_{>0}$-terms $s_1,\dots,s_n$ and linear isosceles terms t_1', t_2' such that $t_1(s_1,\dots,s_n) \overset{e}{\approx} t_1'$, $t_2(s_1,\dots,s_n) \overset{e}{\approx} t_2'$ and $t_1' \overset{se}{\approx} t_2'$.*

Proof. Let s_i be a linear isosceles term with the trace

$$\prod_{\substack{y\in\arg(t)\\ y\neq x_i}} \mathrm{b}(t,y).$$

We may assume that $\arg(s_i)\cap\arg(s_j)=\emptyset$ for $i\neq j$. Then $t_1(s_1,\dots,s_n)$ and $t_2(s_1,\dots,s_n)$ are linear, and moreover for all $x,y\in\bigcup\arg(s_i)$ we have

$$\bar{\mathrm{b}}(t_i(s_1,\dots,s_n),x)=\bar{\mathrm{b}}(t_i(s_1,\dots,s_n),y).$$

Thus, by Corollary 4.3, there exist terms t_1' and t_2' with the desired properties. □

5. $\mathcal{M}$-cancellation and $\mathcal{M}$-polyquasigroups

Recall that a groupoid $(G,\cdot)$ is cancellative if all its left and right translations

$$a\cdot \_ : x\mapsto a\cdot x,\quad \_\cdot b\colon y\mapsto y\cdot b$$

are injective. But some of other types translations may also be interesting.

Example 5.1. Let $(A,+,-,0)$ be a nontrivial abelian group and put $a\cdot b=a-b$. The groupoid $(A,\cdot)$ is an entropic quasigroup, hence cancellative, but its translation $x\mapsto xx$ is not injective.

Example 5.2. Let $k,l>1$ be natural numbers such that $\mathrm{GCD}(k+l,kl)=1$. Then the entropic groupoid $(\mathbb{Z}_{kl},\cdot)$, where $a\cdot b=ka+lb$, is neither left nor right cancellative. Still its translation $x\mapsto xx$ is injective.

These two quite trivial examples show that there are various kinds of cancellativity, which are independent. In this section we develop a language to discuss them.

We distinguish one variable from the set X and denote it by v. Next Lemma is evident but important.

Lemma 5.3. *The set* $\mathrm{Term}(X)$ *has a monoid structure, where multiplication is given by*

$$t(v,x_1,\dots,x_m)\cdot s(v,y_1,\dots,y_n)=t(s(v,y_1,\dots,y_n),x_1,\dots,x_m)$$

and v is its neutral element.

Note that the variable v does not occur in a term t if and only if t is a left-zero element in $(\mathrm{Term}(X),\cdot,v)$, that is iff $ts=t$ for all $s\in\mathrm{Term}(X)$. Let $\mathcal{S}$ be the set of terms in $\mathrm{Term}(X)$ such that v occurs in them, that is

$$\mathcal{S}=\{t\in\mathrm{Term}(X)\mid \mathrm{a}(t,v)\neq 0\}.$$

Note that $(\mathcal{S},\cdot,v)\leqslant(\mathrm{Term}(X),\cdot,v)$.

The following definition is crucial. By a *monoid of (τ-)terms* we mean a subset $\mathcal{M}$ of $\mathrm{Term}(X)$ satisfying

(M1) $(\mathcal{M}, \cdot, v)$ is a submonoid of $(\mathcal{S}, \cdot, v)$;
(M2) $\mathcal{M}$ is closed under substitution: for all $t(v, x_1 \dots, x_n) \in \mathcal{M}$ and $t_1, \dots, t_n \in \mathrm{Term}(X) - \mathcal{S}$ we have $t(v, t_1 \dots, t_n) \in \mathcal{M}$.

Let $\mathcal{L}$ be the set of all terms in $\mathrm{Term}(X)$ such that v occurs in them exactly once and $\mathcal{P}$ be the set of all terms in which v occurs exactly once but always on the rightmost place. One can see that $\mathcal{L}$ and $\mathcal{P}$ are examples of monoids of terms.

The concept of a monoid of terms is very general. In many cases we consider those which satisfy additional conditions. A monoid $\mathcal{M}$ of terms is *proper* provided it satisfies the following condition.

(P) if $\Omega_{>1} \neq \emptyset$ then there exists a term $\eta(x_1, x_2, x_3)$ and distinct variables $y, z \neq v$ such that $\eta(v, y, z), \eta(y, v, z) \in \mathcal{M}$.

Note that x_1, x_2 have to occur in η while x_3 does not. The last condition is auxiliary.

(Ax) For each $\omega \in \Omega_{>1}$ there are an integer $1 \leqslant i \leqslant \tau(\omega)$ and a variable $z \in X$ such that
$$\omega(z, \dots, z, v, z, \dots, z) \in \mathcal{M},$$
where v occurs in the i-th slot;

All $\mathcal{S}, \mathcal{L}, \mathcal{P}$ satisfy (Ax). Moreover $\mathcal{S}$ and $\mathcal{L}$ are proper. But $\mathcal{P}$ is proper only if $\Omega_{>1} = \emptyset$.

A *translation* of an algebra (A, Ω) is a mapping
$$s(\_, a_1, \dots, a_m) \colon A \to A;\ x \mapsto s(x, a_1, \dots, a_m),$$
where $s(v, x_1, \dots, x_m) \in \mathrm{Term}(X)$ and $a_1, \dots, a_m \in A$. Elements $a_1, \dots, a_m$ are called *coefficients* of $s(\_, a_1, \dots, a_m)$. A mapping $f \colon A \to A$ is an $\mathcal{M}$*-translation* if there are a term $s(v, x_1, \dots, x_m) \in \mathcal{M}$ and elements $a_1, \dots, a_m \in A$ such that $f = s(\_, a_1, \dots, a_m)$. Note that $s(\_, a_1, \dots, a_m) = t(\_, b_1, \dots, b_n)$ is possible even if $(s, a_1, \dots, a_m) \neq (t, b_1, \dots, b_n)$. For a subalgebra (B, Ω) of (A, Ω) let
$$T_B^{\mathcal{M}}(A, \Omega) = \{t(\_, b_1, \dots, b_n) \mid t(v, x_1, \dots, x_n) \in \mathcal{M} \text{ and } b_1, \dots, b_n \in B\}$$
be the set of $\mathcal{M}$-translations with coefficients in B.

An algebra is $\mathcal{M}$*-cancellative* if all its $\mathcal{M}$-translations are injective. For instance an algebra is $\mathcal{L}$-cancellative if and only if it is cancellative. If $\mathcal{M}$ is the smallest monoid of groupoid terms containing $v \cdot v$, then the algebra from Example 5.2 is $\mathcal{M}$-cancellative, while the algebra from Example 5.1 is not. We could define $\mathcal{N}$-cancellativity for each subset $\mathcal{N} \subseteq \mathrm{Term}(X)$, but then a given algebra would be $\mathcal{N}$-cancellative iff it is $\langle\mathcal{N}\rangle$-cancellative, where $\langle\mathcal{N}\rangle$ is the smallest monoid of terms containing $\mathcal{N}$. Note that each translation which is not $\mathcal{S}$-translation must be constant. That is why in the definition of a monoid of terms we assumed that $\mathcal{M} \subseteq \mathcal{S}$. The class of all $\mathcal{M}$-cancellative entropic algebras is denoted by $\mathcal{M}$-$\boldsymbol{CE}$.

An algebra is an $\mathcal{M}$*-polyquasigroup* if all its $\mathcal{M}$-translations are bijective. An algebra is a *polyquasigroup* if it is an $\mathcal{L}$-polyquasigroup. The class of all entropic $\mathcal{M}$-polyquasigroups is denoted by $\mathcal{M}$-$\boldsymbol{PE}$.

In the rest of this section we present sample results from the literature about $\mathcal{M}$-polyquasigroups. We would like to convince the reader that, though the concept of $\mathcal{M}$-polyquasigroup is new, special cases were considered in the past.

Recall Evans' generalization of the Bruck-Murdoch-Toyoda theorem for n-quasigroups, that is for polyquasigroups with one n-ary ($n > 1$) basic operation [1, 4].

Theorem 5.4 (Evans' theorem)**.** *If an algebra (A, ω) is a nonempty entropic n-quasigroup, then there exist an abelian group $(A, +, -, 0)$ with n pairwise commuting automorphisms f_i, $i = 1, \ldots, n$, and an element $c \in A$ such that*

$$\omega(a_1, \ldots, a_n) = f_1(a_1) + \ldots + f_n(a_n) + c.$$

An algebra (A, Ω) is called *permutational* if each of its translations is either a permutation or a constant. Note that each $\mathcal{S}$-polyquasigroup is permutational. The next theorem, proved by P. Pálfy in [18], plays an important role in tame congruence theory (see [7]).

Theorem 5.5 (Pálfy's theorem)**.** *Let (A, Ω) be a permutational algebra with at least three elements. Assume that there exists a natural number m such that for all elements $a, b \in A$ the class $a/\operatorname{cg}(a, b)$ of the principal congruence generated by the pair (a, b) has cardinality not greater than m. Then the algebra (A, Ω) is polynomially equivalent to a certain unary algebra or to a vector space over a finite field.*

A permutational entropic algebra satisfying the condition of Pálfy's theorem, and possessing a polynomial operation which depends on two variables, is strongly entropic (see Remark 9.4). For instance, a permutational algebra (A, ω) with one basic operation, satisfying the condition of Pálfy's theorem and possessing a polynomial operation which depends on two variables, is entropic, and thus strongly entropic. This fact follows also from Proposition 3.4.

6. Generating $\boldsymbol{SE}$

Here we prove the following theorem.

Theorem 6.1. *Let $\mathcal{M}$ be a proper monoid of terms satisfying* (Ax)*. Then the class $\mathcal{M}$-$\boldsymbol{CE}$ generates the variety $\boldsymbol{SE}$.*

At the end of the paper we will show that the condition (Ax) is irrelevant here.

Let $\overset{\mathcal{M}}{\approx}$ be the equational theory of $\mathcal{M}$-$\boldsymbol{CE}$. We will show that, under the assumptions of Theorem 6.1, this theory coincides with $\overset{se}{\approx}$. The proof is divided into several steps.

Lemma 6.2. *Let $t(x, y, z)$ be an isosceles term without constants and linear relative to x and y. If $\bar{\mathrm{a}}(t, x) = \bar{\mathrm{a}}(t, y)$ then $t(x, y, z) \overset{e}{\approx} t(y, x, z)$.*

Proof. Assume that $x \neq y$. We proceed by induction on the depth of t. If the assumption is satisfied then the depth cannot be smaller than two. Indeed, if the depth of t is equal to 0 or 1 and $\bar{\mathrm{a}}(t, x) = \bar{\mathrm{a}}(t, y)$ then $\mathrm{a}(t, x) = \mathrm{a}(t, y)$ and hence $x = y$. If the depth is equal to two then the required identity is an immediate consequence of entropicity. Assume that the depth is at least three. Let $\mathrm{a}(t,x)=\alpha$ and $\mathrm{a}(t,y)=\beta$. If $\alpha=(\omega,i)\alpha'$ and $\beta=(\omega,i)\beta'$ then

$$t(x,y,z)=\omega\ (s_1(z),\dots,s_{i-1}(z),s_i(x,y,z),s_{i+1}(z),\dots,s_{\tau(\omega)}(z))$$

and $\mathrm{a}(s_i,x)=\alpha'$, $\mathrm{a}(s_i,y)=\beta'$. The term s_i satisfies the assumptions of Lemma and its depth is smaller than the depth of t. Hence $s_i(x,y,z)\overset{e}{\approx}s_i(y,x,z)$ and immediately $t(x,y,z)\overset{e}{\approx}t(y,x,z)$. Now let $\alpha=(\omega,i)\alpha'$, $\beta=(\omega,j)\beta'$ and $i\neq j$. Because t is isosceles there exists an isosceles term $t'(x,y,z)$ linear relative to x and y such that $\mathrm{a}(t',x)=(\omega,i)(\omega,j)\alpha''$ and $\mathrm{a}(t',y)=(\omega,j)(\omega,i)\beta''$ and $t\overset{e}{\approx}t'$. If $\alpha''=\beta''$ then, by entropicity, we have $t'(x,y,z)\overset{e}{\approx}t'(y,x,z)$. Now assume that $\alpha''\neq\beta''$. Then there exists an isosceles term $t''(x,y,x',y',z)$ linear relative to x,x',y,y' such that $t''(x,y,z,z,z)=t'(x,y,z)$ and moreover $\mathrm{a}(t'',x')=(\omega,i)(\omega,j)\beta''$, $\mathrm{a}(t'',y')=(\omega,j)(\omega,i)\alpha''$. Then

$$t(x,y,z)\overset{e}{\approx}t''(x,y,z,z,z)\overset{e}{\approx}t''(z,z,y,x,z)\overset{e}{\approx}t''(y,x,z,z,z)\overset{e}{\approx}t(y,x,z).$$

The second equality follows from entropicity and the third follows from the inductive assumption. □

Lemma 6.3. *Let $\mathcal{M}$ be a proper monoid of terms and $t(x_1,x_2,\dots,x_n,z)$ be an isosceles $\tau_{>0}$-term linear relative to x_1 and x_2. If $\bar{\mathrm{a}}(t,x_1)=\bar{\mathrm{a}}(t,x_2)$, then*

$$t(x_1,x_2,x_3,\dots,x_n,z)\overset{\mathcal{M}}{\approx}t(x_2,x_1,x_3,\dots,x_n,z).$$

Proof. We may assume that $\Omega_{>1}\neq\emptyset$. We prove the assertion by induction on n. For $n=2$ it follows from previous Lemma. So let $n>2$ and assume that the assertion is true for $n-1$. The monoid $\mathcal{M}$ of terms is proper so there exist a term η as in the condition (P). Note that without lost of generality we may assume that η is a symbol of a basic operation. Indeed, let $\sigma=\tau\cup\{(\mu,3)\}$, and $\mathcal{N}$ be a monoid of σ-terms generated by $\mathcal{M}\cup\{\mu(v,y,z),\mu(v,y,z)\}$. Then $\mathcal{M}$-$\boldsymbol{CE}$ is equivalent to the quasivariety of $\mathcal{N}$-cancellative entropic σ-algebras satisfying the identity $\eta(x,y,z)=\mu(x,y,z)$. Let $s(y_1,\dots,y_{n-1},z)$ be an isosceles term without constants such that $\bar{\mathrm{a}}(s,y_k)=(\eta,1)^p(\eta,2)^q$ for all k and some natural numbers p,q. Then the term $s(v,y_2,\dots,y_{n-1},z)$ belongs to $\mathcal{M}$. By induction, we have

$$s(y_1,\dots,y_{n-1},z)\overset{\mathcal{M}}{\approx}s(y_{\sigma(1)},\dots,y_{\sigma(n-1)},z)\tag{6.1}$$

for each permutation σ of the set $\{1,\dots,n-1\}$. Hence

$$s\big(t(x_1,x_2,x_3,\dots,x_n,z),t(z,z,x_4,\dots,x_n,z,x_3),$$
$$t(z,z,x_5,\dots,x_n,z,x_3,x_4),\dots,t(z,z,z,x_3,\dots,x_n),t(z,\dots,z)\big)$$
$$\overset{e}{\approx} t\big(s(x_1,z,\dots,z),s(x_2,z,\dots,z),s(x_3,\dots,x_n,z,z),$$
$$s(x_4,\dots,x_n,z,x_3,z),\dots,s(x_n,z,x_3,\dots,x_{n-1},z),s(z,x_3,\dots,x_n,z)\big)$$
$$\overset{\mathcal{M}}{\approx} t\big(s(x_2,z,\dots,z),s(x_1,z,\dots,z),s(x_3,\dots,x_n,z,z),$$
$$s(x_4,\dots,x_n,z,x_3,z),\dots,s(x_n,z,x_3,\dots,x_{n-1},z),s(z,x_3,\dots,x_n,z)\big)$$
$$\overset{e}{\approx} s\big(t(x_2,x_1,x_3\dots,x_n,z),t(z,z,x_4,\dots,x_n,z,x_3),$$
$$t(z,z,x_5,\dots,x_n,z,x_3,x_4),\dots,t(z,z,z,x_3,\dots,x_n),t(z,\dots,z)\big).$$

The first and third identities follow directly from entropicity and the second from (6.1) and Lemma 6.2. Finally, by the $\mathcal{M}$-cancellativity, we have

$$t(x_1,x_2,x_3,\dots,x_n,z)\overset{\mathcal{M}}{\approx} t(x_2,x_1,x_3,\dots,x_n,z).$$

□

Corollary 6.4. *Let $\mathcal{M}$ be a proper monoid of terms. If $t(x_1,\dots,x_n)$ is an isosceles linear $\tau_{>0}$-term and σ is a permutation of the set $\{1,\dots,n\}$ such that for all i we have $\bar{a}(t,x_i)=\bar{a}(t,x_{\sigma(i)})$ then $t(x_1,\dots,x_n)\overset{\mathcal{M}}{\approx} t(x_{\sigma(1)},\dots,x_{\sigma(n)})$.*

Lemma 6.5. *Let $\mathcal{M}$ be a proper monoid of terms and $t_1(x_1,\dots,x_n)$, $t_2(x_1,\dots,x_n)$ be isosceles linear terms such that $t_1\overset{se}{\approx} t_2$. Then $t_1\overset{\mathcal{M}}{\approx} t_2$.*

Proof. There are two easy cases to consider. First, concerns the case when terms t_1, t_2 are constant, which is trivial, and second when there are no constants occurring in t_1 and t_2. Assume that the second case holds. Let γ be the trace of t_1 and γ' be the trace of t_2. Then γ and γ' differ only by the order of their letters. Hence there is an isosceles linear term (without constants) $t_2'(x_1,\dots,x_n)$ with the trace γ such that $t_2\overset{e}{\approx} t_2'$. In fact

$$t_2'(x_1,\dots,x_n)=t_1(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

for some permutation σ. Thus, this case follows from Corollary 6.4.

Now we consider a general case. As above we may assume that t_1 and t_2 have the same trace γ. One can prove by induction on the depth of considered terms that if p_1 and p_2 are two isosceles terms of the same trace then

$$\sum_{y\in X\cup\Omega_0}\bar{a}(p_1,y)=\sum_{y\in X\cup\Omega_0}\bar{a}(p_2,y).$$

Applying cancellativity of the semigroup $(\mathbb{N}[\Sigma],+)$ and the above equality for terms t_1 and t_2 we get

$$\bar{a}(t_1,o)=\bar{a}(t_2,o).$$

Thus there exist linear isosceles terms $t'_k(x_1,\dots,x_n,y_1,\dots,y_m)$ without constants such that $t_k(x_1,\dots,x_n)=t'_k(x_1,\dots,x_n,o,\dots,o)$ and $\bar{\mathrm{a}}(t'_1,y_j)=\bar{\mathrm{a}}(t'_2,y_j)$ for all j. But terms t'_1 and t'_2 falls into the previous case what finishes the proof. □

Lemma 6.6. *Let $\mathcal{M}$ be a proper monoid of terms, $t_1(x_1,\dots,x_n)$ and $t_2(x_1,\dots,x_n)$ be two linear terms and assume that $t_1 \overset{se}{\approx} t_2$. There exist linear isosceles $\tau_{>0}$-terms $s_1,\dots,s_n$ with mutually disjoint sets of variables such that*

$$t_1(s_1,\dots,s_n) \overset{\mathcal{M}}{\approx} t_2(s_1,\dots,s_n).$$

Proof. By Proposition 4.4 and Lemma 6.5. □

Lemma 6.7. *Let $\mathcal{M}$ be a proper monoid of terms satisfying* (Ax). *Let $t_1(x_1,\dots,x_n)$ and $t_2(x_1,\dots,x_n)$ be two linear terms such that $t_1 \overset{se}{\approx} t_2$. Then $t_1 \overset{\mathcal{M}}{\approx} t_2$.*

Proof. By Lemma 6.6, there exist linear $\tau_{>0}$-terms s_i, with mutually disjoint sets of variables, such that $t_1(s_1,\dots,s_n) \overset{\mathcal{M}}{\approx} t_2(s_1,\dots,s_n)$. By downward induction on j, we prove that

$$t_1(s_1,\dots,s_{j-1},x_j,\dots,x_n) \overset{\mathcal{M}}{\approx} t_2(s_1,\dots,s_{j-1},x_j,\dots,x_n). \tag{6.2}$$

For $j=n+1$ it is already proved. Assume that $t_1(s_1,\dots,s_j,x_{j+1},\dots,x_n) \overset{\mathcal{M}}{\approx} t_2(s_1,\dots,s_j,x_{j+1},\dots,x_n)$. Let $t_k(s_1,\dots,s_{j-1},x,x_{j+1},\dots,x_n)=t'_k(y_1,\dots,y_l,x)$ and $s_j=s$. The conditions (M1), (M2) and (Ax) imply that there are a natural number i and a variable $z\in X$ such that $s(z,\dots,z,v,z,\dots,z)\in\mathcal{M}$, where v is in the i-th slot. For simplicity we may assume that $i=1$. Now let us compute

$$\begin{aligned}
&s(t'_1(y_1,\dots,y_l,x),t'_1(y_1,\dots,y_l,s(x,\dots,x)),\dots,t'_1(y_1,\dots,y_l,s(x,\dots,x)))\\
\overset{e}{\approx}\;& t'_1(s(y_1,\dots,y_1),\dots,s(y_l,\dots,y_l),s(x,s(x,\dots,x),\dots,s(x,\dots,x)))\\
\overset{\mathcal{M}}{\approx}\;& t'_2(s(y_1,\dots,y_1),\dots,s(y_l,\dots,y_l),s(x,s(x,\dots,x),\dots,s(x,\dots,x)))\\
\overset{e}{\approx}\;& s(t'_2(y_1,\dots,y_l,x),t'_2(y_1,\dots,y_l,s(x,\dots,x)),\dots,t'_2(y_1,\dots,y_l,s(x,\dots,x)))\\
\overset{\mathcal{M}}{\approx}\;& s(t'_2(y_1,\dots,y_l,x),t'_1(y_1,\dots,y_l,s(x,\dots,x)),\dots,t'_1(y_1,\dots,y_l,s(x,\dots,x))).
\end{aligned}$$

And by the $\mathcal{M}$-cancellativity we obtain (6.2). In the case $j=1$ this is the required claim. □

Proof of Theorem 6.1. Note that Proposition 3.2 and Lemma 6.7 imply the inclusion $\overset{se}{\approx}\subseteq\overset{\mathcal{M}}{\approx}$. To prove the converse it is enough to find an $\mathcal{M}$-cancellative entropic algebra satisfying only the identities from $\overset{se}{\approx}$. Let $(\mathbb{Z}[\Sigma],+,-,0,\cdot,1)$ be the integral domain of polynomials with commuting indeterminants from Σ and integer coefficients. Let $(K,+,-,0,\cdot,1)$ be the field of quotients of $(\mathbb{Z}[\Sigma],+,-,0,\cdot,1)$. Consider the vector space $(V(X),+,-,0,K)$ over $(K,+,-,0,\cdot,1)$ with linear basis X. Let $(V(X),\Omega)$ be the τ-algebra with basic operations given by

$$\omega(m_1,\dots,m_{\tau(\omega)})=(\omega,1)m_1+\ldots+(\omega,\tau(\omega))m_{\tau(\omega)}$$

for $\omega \in \Omega_{>0}$ and

$$o = 0$$

if $\Omega_0 \neq \emptyset$. As a reduct of a vector space, this algebra is strongly entropic. Moreover, by Proposition 3.1, it contains a free strongly entropic algebra over X as a subalgebra. Thus it cannot satisfy identities that are not in $\overset{se}{\approx}$. Finally, $(V(X), \Omega)$ is an $\mathcal{S}$-polyquasigroup. Indeed, let $t(v, x_1, \ldots, x_n) \in \mathcal{S}$, $a_1, \ldots, a_n, b \in V(X)$ and consider the equation

$$t(x, a_1, \ldots, a_n) = b.$$

It has the unique solution given by

$$x = \bar{\mathrm{a}}(t, v)^{-1}\big(b - \sum_{i=1}^{n} \bar{\mathrm{a}}(t, x_i) a_i\big).$$

□

7. Embedding into $\mathcal{M}$-polyquasigroups

Here we prove that each $\mathcal{M}$-cancellative strongly entropic algebra embeds into a strongly entropic $\mathcal{M}$-polyquasigroup. We obtain prevailing part of the proof by translating its groupoid case from [9, Chapter 5] into an sufficiently general language. We start with some definitions.

Let $\boldsymbol{D}$ be a full subcategory of a category $\boldsymbol{C}$. A functor $R\colon \boldsymbol{C} \to \boldsymbol{D}$ is called a *reflector* if it is left adjoint to the inclusion functor $J\colon \boldsymbol{D} \to \boldsymbol{C}$. For the existence of a reflector $R\colon \boldsymbol{C} \to \boldsymbol{D}$ it is necessary and sufficient that for each $c \in \boldsymbol{C}$ there is an object $R(c) \in \boldsymbol{D}$ and a morphism $r_c\colon c \to R(c)$ such that for each morphism $f\colon c \to d$, where $d \in \boldsymbol{D}$, there is exactly one morphism $\tilde{f}\colon R(c) \to d$ with $f = \tilde{f} \circ r_c$. We say that the object $R(c)$ is the *reflection of c in $\boldsymbol{D}$* and the morphism r_c is the *reflecting morphism of c into $\boldsymbol{D}$*. We are interested in categories of algebras where morphisms are homomorphisms. In this case we say that a reflector $R\colon \boldsymbol{C} \to \boldsymbol{D}$ is *injective* if all corresponding reflecting homomorphisms r_c, where $c \in \boldsymbol{C}$, are injective. For more information about reflectors, see [15, Chapter IV].

The class of all $\mathcal{M}$-cancellative strongly entropic algebras is denoted by $\mathcal{M}$-$\boldsymbol{CSE}$ and the class of all strongly entropic $\mathcal{M}$-polyquasigroups is denoted by $\mathcal{M}$-$\boldsymbol{PSE}$.

Theorem 7.1. *For a monoid $\mathcal{M}$ of terms there exists an injective reflector from the category of $\mathcal{M}$-cancellative strongly entropic algebras into the category of strongly entropic $\mathcal{M}$-polyquasigroups*

$$R^{\mathcal{M}}\colon \mathcal{M}\text{-}\boldsymbol{CSE} \to \mathcal{M}\text{-}\boldsymbol{PSE}.$$

In particular, a strongly entropic $\mathcal{M}$-cancellative algebra embeds into a strongly entropic $\mathcal{M}$-polyquasigroup. Moreover for $(A, \Omega) \in \mathcal{M}$-$\boldsymbol{CSE}$:

(1) *(A, Ω) satisfies precisely the same identities as $R^{\mathcal{M}}(A, \Omega)$;*

(2) *if (A, Ω) is subdirectly irreducible, then $R^{\mathcal{M}}(A, \Omega)$ is subdirectly irreducible; and if additionally (A, Ω) is a mode, then*

(3) (A, Ω) *and* $R^{\mathcal{M}}(A, \Omega)$ *satisfy precisely the same first order universal sentences;*
(4) *if* (A, Ω) *is simple, then* $R^{\mathcal{M}}(A, \Omega)$ *is simple.*

The proof of Theorem 7.1 will be carried out in several steps.

Consider translations $f = s(\_, a_1, \ldots, a_m)$ and $g = t(\_, b_1, \ldots, b_n)$ in $T_B^{\mathcal{M}}(A, \Omega)$. By f^g we denote a translation in $T_B^{\mathcal{M}}(A, \Omega)$ given by

$$f^g = s(\_, t(a_1, d, \ldots, d), \ldots, t(a_m, d, \ldots, d)), \tag{7.1}$$

where d is a certain fixed element in B. Obviously, the translation f^g is not uniquely determined because it depends on the choice of terms s, t, coefficients a_i, b_j and element d. Similar situations happen few times in this section. Fortunately, this will not cause any problem, because what we need in proofs is only the existence of translations with certain properties. The uniqueness is not necessary. Now note that if the algebra (A, Ω) is entropic then

$$\begin{aligned} f^g \circ g &= s(t(\_, b_1, \ldots, b_n), t(a_1, d, \ldots, d), \ldots, t(a_m, d, \ldots, d)) \\ &= t(s(\_, a_1, \ldots, a_m), s(b_1, d, \ldots, d), \ldots, s(b_n, d, \ldots, d)) = g^f \circ f. \end{aligned}$$

In other words, given f_1 and f_2 in $T_B^{\mathcal{M}}(A, \Omega)$ one can find g_1 and g_2 in $T_B^{\mathcal{M}}(A, \Omega)$ such that $g_1 \circ f_1 = g_2 \circ f_2$ Thus, we get the following crucial lemma.

Lemma 7.2. *Let* (A, Ω) *be an entropic algebra with a subalgebra* (B, Ω)*, and* $f_1, \ldots, f_k \in T_B^{\mathcal{M}}(A, \Omega)$*. Then there exist* $g_1, \ldots, g_k \in T_B^{\mathcal{M}}(A, \Omega)$ *such that*

$$g_1 \circ f_1 = \ldots = g_k \circ f_k.$$

For a given translation $f = t(\_, a_1, \ldots, a_n) \in T_B^{\mathcal{M}}(A, \Omega)$ and a term $s(x_1, \ldots, x_l)$ we define a new translation $\overset{s}{f}$ from $T_B^{\mathcal{M}}(A, \Omega)$ by

$$\overset{s}{f} = t(\_, s(a_1, \ldots, a_1), \ldots, s(a_n, \ldots, a_n)).$$

Now if (A, Ω) is entropic, then

$$s(f(b_1), \ldots, f(b_l)) = \overset{s}{f}(s(b_1, \ldots, b_l)).$$

In a case when $s = \omega(x_1, \ldots, x_{\tau(\omega)})$ we write $\overset{\omega}{f}$ instead of $\overset{s}{f}$.

A subalgebra (B, Ω) of an algebra (A, Ω) is $\mathcal{M}$*-closed* if $f^{-1}(B) \subseteq B$ for all $f \in T_B^{\mathcal{M}}(A, \Omega)$.

Lemma 7.3. *Let* (A, Ω) *be an entropic algebra with* (B, Ω) *as subalgebra. Then the smallest* $\mathcal{M}$*-closed subalgebra of* (A, Ω) *containing* B *is given by*

$$C = \{\, a \in A \mid f(a) \in B \text{ for some } f \in T_B^{\mathcal{M}}(A, \Omega)\}.$$

Proof. It is obvious that $B \subseteq C$ and that each $\mathcal{M}$-closed subalgebra of (A, Ω) containing B must contain C as well. So it is enough to show that C is an $\mathcal{M}$-closed subalgebra of (A, Ω). Let $\omega \in \Omega_{>0}$, $a_1, \ldots, a_{\tau(\omega)} \in C$ and $f_1, \ldots, f_{\tau(\omega)} \in T_B^{\mathcal{M}}(A, \Omega)$

be such that $f_i(a_i) \in B$ for $i \in \{1, \dots, \tau(\omega)\}$. By Lemma 7.2, we may assume that $f_1 = \dots = f_{\tau(\omega)} = f$. Then

$$\overset{\omega}{f}(\omega(a_1, \dots, a_{\tau(\omega)}) = \omega(f(a_1), \dots, f(a_{\tau(\omega)})) \in B.$$

Thus, C forms a subalgebra.

Now assume that $g(a) \in C$ for some $g \in T_C^{\mathcal{M}}(A, \Omega)$. This means that there is $h \in T_B^{\mathcal{M}}(A, \Omega)$ such that $h \circ g(a) \in B$. Clearly, the translation $k = h \circ g$ belongs to $T_C^{\mathcal{M}}(A, \Omega)$. Assume that

$$k = t(\_, c_1, \dots, c_n),$$

where $c_1, \dots, c_n \in C$. Let $f \in T_B^{\mathcal{M}}(A, \Omega)$ be such that $f(c_1), \dots, f(c_n) \in B$. Then $t(\_, f(c_1), \dots, f(c_n)) \circ f \in T_B^{\mathcal{M}}(A, \Omega)$ and

$$t(f(a), f(c_1), \dots, f(c_n)) = \overset{t}{f}(t(a, c_1, \dots, c_n)) = \overset{t}{f} \circ k(a) \in B.$$

This proves $\mathcal{M}$-closedness. □

A subalgebra (B, Ω) of (A, Ω) is $\mathcal{M}$-*dense* if the smallest $\mathcal{M}$-closed subalgebra containing B is (A, Ω). A congruence θ of an algebra (A, Ω) is $\mathcal{M}$-*cancellative* if $a \,\theta\, b$ whenever $f(a) \,\theta\, f(b)$ for some $f \in T_A^{\mathcal{M}}(A, \Omega)$. This condition is equivalent to the statement that the algebra $(A/\theta, \Omega)$ is $\mathcal{M}$-cancellative.

Lemma 7.4. *Assume that (B, Ω) is an $\mathcal{M}$-dense subalgebra of (A, Ω) and that θ is an $\mathcal{M}$-cancellative congruence on (B, Ω). Then there is exactly one $\mathcal{M}$-cancellative congruence $\tilde{\theta}$ on (A, Ω) extending θ.*

Proof. Existence: Define a relation $\tilde{\theta} \subseteq A^2$ as follows

$$a \,\tilde{\theta}\, b \iff f(a) \,\theta\, f(b) \text{ for some } f \in T_B^{\mathcal{M}}(A, \Omega).$$

We show that $\tilde{\theta}$ is an $\mathcal{M}$-cancellative congruence extending θ. Reflexivity of $\tilde{\theta}$ follows from $\mathcal{M}$-density of (B, Ω) in (A, Ω). Symmetry is obvious and transitivity follows from Lemma 7.2 and transitivity of θ. Now let $\omega \in \Omega$ and $f(a_i) \;\theta\; f(b_i)$, where $f \in T_B^{\mathcal{M}}(A, \Omega)$ and $a_i, b_j \in A$. Then

$$\begin{aligned} \overset{\omega}{f}(\omega(a_1, \dots, a_{\tau(\omega)})) &= \omega(f(a_1), \dots, f(a_{\tau(\omega)})) \\ &\theta\; \omega(f(b_1), \dots, f(b_{\tau(\omega)})) = \overset{\omega}{f}(\omega(b_1, \dots, b_{\tau(\omega)})). \end{aligned}$$

This shows that $\tilde{\theta}$ is a congruence of (A, Ω). To show that $\tilde{\theta}$ is $\mathcal{M}$-cancellative one can use similar technique. Finally the equality $\tilde{\theta} \cap B^2 = \theta$ follows from the $\mathcal{M}$-cancellativity of θ.

Uniqueness: Let θ_1 and θ_2 be two $\mathcal{M}$-cancellative congruences on (A, Ω) extending θ. Assume that $a \,\theta_1\, b$. Let f be a translation in $T_B^{\mathcal{M}}(A, \Omega)$ such that $f(a), f(b) \in B$. Then $f(a) \,\theta\, f(b)$ and hence $f(a) \,\theta_2\, f(b)$. By the $\mathcal{M}$-cancellativity we have $a \,\theta_2\, b$. □

Proposition 7.5. *Let (A, Ω) be an entropic $\mathcal{M}$-polyquasigroup with (B, Ω) as an $\mathcal{M}$-dense subalgebra. Then (A, Ω) is a reflection of (B, Ω) in $\mathcal{M}$-**PE**, with the embedding $A \hookrightarrow B$ as the corresponding reflecting homomorphism.*

Proof. Let $h\colon (B,\Omega) \to (C,\Omega)$, where $(C,\Omega) \in \mathcal{M}$-**PE**. We need to show that h may be extended to a homomorphism $\tilde{h}\colon (A,\Omega) \to (C,\Omega)$. First we define the mapping

$$_{-h}\colon\ T_B^{\mathcal{M}}(A,\Omega) \to T_{h(B)}^{\mathcal{M}}(C,\Omega);\ \ r(\_,b_1,\dots,b_n) \mapsto r(\_,h(b_1),\dots,h(b_n)).$$

Then

$$f_h \circ h = h \circ f|_B.$$

Now we define $\tilde{h}$. For $a \in A$ let $f \in T_B^{\mathcal{M}}(A,\Omega)$ be such that $f(a) = b \in B$ and put

$$\tilde{h}(a) = f_h^{-1} \circ h(b).$$

Claim 1. $\tilde{h}$ is well defined.
Recall that in the definition of f^g or g^f given by (7.1) we used a fixed element $d \in B$. Now choose $e = h(d) \in C$ and use this element in the definition of translations ${g_h}^{f_h}$ and ${f_h}^{g_h}$ in $T_{h(B)}^{\mathcal{M}}(C,\Omega)$. Then

$$\begin{aligned}
{g_h}^{f_h} \circ f_h \circ g_h^{-1} \circ h &= {f_h}^{g_h} \circ g_h \circ g_h^{-1} \circ h\\
&= {f_h}^{g_h} \circ h = (f^g)_h \circ h = h \circ f^g|_B\\
&= h \circ f^g \circ g \circ g^{-1}|_B = h \circ g^f \circ f \circ g^{-1}|_B\\
&= (g^f)_h \circ h \circ f \circ g^{-1}|_B = {g_h}^{f_h} \circ h \circ f \circ g^{-1}|_B.
\end{aligned}$$

Hence by injectivity of $g_h^{f_h}$

$$f_h \circ g_h^{-1} \circ h = h \circ f \circ g^{-1}|_B$$

and

$$g_h^{-1} \circ h \circ g|_{g^{-1}(B)} = f_h^{-1} \circ h \circ f \circ g^{-1} \circ g|_{g^{-1}(B)} = f_h^{-1} \circ h \circ f|_{g^{-1}(B)}.$$

Thus, $\tilde{h}$ well defined.
Claim 2. $\tilde{h}$ is a homomorphism.
Let $\omega \in \Omega$, $a_1,\dots,a_{\tau(\omega)} \in A$ and let $f \in T_B^{\mathcal{M}}(A,\Omega)$ be such that $f(a_i) = b_i \in B$ for all i. Then $\overset{\omega}{f}(\omega(a_1,\dots,a_{\tau(\omega)})) \in B$. Further

$$\begin{aligned}
\overset{\omega}{f}_h(\omega(\tilde{h}(a_1),\dots,\tilde{h}(a_{\tau(\omega)}))) &= \omega(f_h \circ \tilde{h}(a_1),\dots,f_h \circ \tilde{h}(a_{\tau(\omega)}))\\
&= \omega(f_h \circ f_h^{-1} \circ h(b_1),\dots,f_h \circ f_h^{-1} \circ h(b_{\tau(\omega)}))\\
&= \omega(h(b_1),\dots,h(b_{\tau(\omega)}))\\
&= h(\omega(b_1,\dots,b_{\tau(\omega)}))\\
&= h(\omega(f(a_1),\dots,f(a_{\tau(\omega)})))\\
&= h(\overset{\omega}{f}(\omega(a_1,\dots,a_{\tau(\omega)})))\\
&= \overset{\omega}{f}_h \circ (\overset{\omega}{f}_h)^{-1} \circ h(\overset{\omega}{f}(\omega(a_1,\dots,a_{\tau(\omega)})))\\
&= \overset{\omega}{f}_h \circ \tilde{h}(\omega(a_1,\dots,a_{\tau(\omega)}).
\end{aligned}$$

Now by injectivity of $\overset{\omega}{f}_h$ we have

$$\omega(\tilde{h}(a_1), \dots, \tilde{h}(a_{\tau(\omega)})) = \tilde{h}(\omega(a_1, \dots, a_{\tau(\omega)})).$$

Claim 3. $\tilde{h}$ is unique.
Let $k\colon (A, \Omega) \to (C, \Omega)$ be a homomorphism extending h. Let $a \in A$ and let $f \in T_B^{\mathcal{M}}(A, \Omega)$ be a translation such that $f(a) = b \in B$. We have

$$f_k \circ k = k \circ f$$

and hence

$$k \circ f^{-1} = f_k^{-1} \circ k.$$

Thus,

$$k(a) = k \circ f^{-1}(b) = f_k^{-1} \circ k(b) = f_h^{-1} \circ h(b) = \tilde{h}(a).$$

It is obvious that $\tilde{h}$ extends h, so this finishes the proof. □

Lemma 7.6. *Let (A, Ω) be an $\mathcal{M}$-cancellative entropic algebra and consider a translation $f = r(\_, c_1, \dots, c_k)$ in $T_A^{\mathcal{M}}(A, \Omega)$. If t_1 and t_2 are terms such that $t_1(c_i, \dots, c_i) = t_2(c_i, \dots, c_i)$ for all $i \in \{1, \dots, k\}$, then*

$$t_1(a_1, \dots, a_m) = t_2(b_1, \dots, b_n) \iff t_1(f(a_1), \dots, f(a_m)) = t_2(f(b_1), \dots, f(b_n)).$$

Proof. By the assumption $\overset{t_1}{f} = \overset{t_2}{f}$ and this mapping is 1-1. Thus

$$\overset{t_1}{f}(t_1(a_1, \dots, a_m)) = \overset{t_2}{f}(t_2(b_1, \dots, b_n)) \iff t_1(a_1, \dots, a_m) = t_2(b_1, \dots, b_n)$$

as required. □

Proposition 7.7. *Let (A, Ω) be an $\mathcal{M}$-cancellative entropic algebra with (B, Ω) as an $\mathcal{M}$-dense subalgebra. Then both algebras satisfy the same identities.*

Proof. Assume that (B, Ω) satisfies $t_1(x_1, \dots, x_m) \approx t_2(y_1, \dots, y_n)$. Then obviously (B, Ω) satisfies $t_1(x, \dots, x) \approx t_2(x, \dots, x)$ as well. Let $a_1, \dots, a_m, b_1, \dots, b_n \in A$. Let $f \in T_B^{\mathcal{M}}(A, \Omega)$ be a translation such that $f(a_i) \in B$ and $f(b_j) \in B$ for all i, j. Then, by the assumption, we have

$$t_1(f(a_1), \dots, f(a_m)) = t_2(f(b_1), \dots, f(b_n)).$$

Now by Lemma 7.6,

$$t_1(a_1, \dots, a_m) = t_2(b_1, \dots, b_n).$$

Thus (A, Ω) satisfies $t_1(x_1, \dots, x_m) \approx t_2(y_1, \dots, y_n)$. The converse is evident. □

Proposition 7.8. *Let (A, Ω) be an $\mathcal{M}$-cancellative entropic algebra with (B, Ω) as an $\mathcal{M}$-dense idempotent subalgebra. Then both algebras satisfy the same first order universal sentences. In particular they satisfy the same quasi-identities.*

Proof. It is enough to consider a sentence $\varphi = \forall x_1, \dots, x_n\ \psi$, where ψ is a quantifier-free formula of the form

$$\psi = \bigvee_i \bigwedge_j \chi_j^i,$$

and χ^i_j are atomic or negated atomic formulas. Let $a_1, \dots, a_n \in A$. We would like to show that if $(B, \Omega) \models \varphi$, then $(A, \Omega) \models \psi(a_1, \dots, a_n)$. Choose $f \in T^{\mathcal{M}}_B(A, \Omega)$ such that $f(a_i) \in B$ for all i. We have the following equivalences

$$\begin{aligned} (A,\Omega) \models \psi(a_1,\dots,a_n) \quad &\text{iff} \quad \exists i\ \forall j \quad (A,\Omega) \models \chi^i_j(a_1,\dots,a_n) \\ &\text{iff} \quad \exists i\ \forall j \quad (A,\Omega) \models \chi^i_j(f(a_1),\dots,f(a_n)) \\ &\text{iff} \quad \exists i\ \forall j \quad (B,\Omega) \models \chi^i_j(f(a_1),\dots,f(a_n)) \\ &\text{iff} \quad (B,\Omega) \models \psi(f(a_1),\dots,f(a_n)). \end{aligned}$$

The second "iff" follows from Lemma 7.6 and the last sentence is true according to the assumption. Thus (A, Ω) satisfies φ. The converse is clear. □

Proposition 7.9. *Let (A, Ω) be an $\mathcal{M}$-cancellative entropic algebra with (B, Ω) as an $\mathcal{M}$-dense subalgebra. If (B, Ω) is subdirectly irreducible then so is (A, Ω).*

Proof. Denote by $\mathrm{cg}_{(C,\Omega)}(c_1, c_2)$ the congruence on (C, Ω) generated by the pair (c_1, c_2). Assume that (B, Ω) is subdirectly irreducible and that $\mathrm{cg}_{(B,\Omega)}(b_1, b_2)$ is the least nontrivial congruence on (B, Ω). Let a_1 and a_2 be distinct elements in A and $f \in T^{\mathcal{M}}_B(A, \Omega)$ be such that $f(a_1), f(a_2) \in B$. But f is 1-1, hence $f(a_1) \neq f(a_2)$ and $(b_1, b_2) \in \mathrm{cg}_{(B,\Omega)}(f(a_1), f(a_2))$. Thus $(b_1, b_2) \in \mathrm{cg}_{(A,\Omega)}(a_1, a_2)$. This means that $\mathrm{cg}_{(A,\Omega)}(b_1, b_2)$ is the least nontrivial congruence on (A, Ω). □

Proposition 7.10. *Let (A, Ω) be an entropic $\mathcal{M}$-polyquasigroup with (B, Ω) as an $\mathcal{M}$-dense idempotent subalgebra. If (B, Ω) is simple then so is (A, Ω).*

Proof. Let θ be a nontrivial congruence on (A, Ω) and a_1, a_2 be distinct elements such that $a_1 \,\theta\, a_2$. Because there is an $f \in T^{\mathcal{M}}_B(A, \Omega)$ with $f(a_1), f(a_2) \in B$ and (B, Ω) is simple, we have $B^2 \subseteq \theta$. Next note that for $f \in T^{\mathcal{M}}_B(A, \Omega)$ there is an isomorphism $f|_{f^{-1}(B)} \colon (f^{-1}(B), \Omega) \to (B, \Omega)$ and that $B \subseteq f^{-1}(B)$. Thus $f^{-1}(B)^2 \subseteq \theta$. Finally

$$A^2 = \bigcup_{f \in T^{\mathcal{M}}_B(A,\Omega)} f^{-1}(B)^2 \subseteq \theta.$$

□

Proof of Theorem 7.1. For a given set U, let $(V(U), \Omega)$ be an algebra defined as in the proof of Theorem 6.1 in the case $U = X$. Note that $(P(U), \Omega)$, the free strongly entropic algebra over U, coincide with the subalgebra of $(V(U), \Omega)$ generated by the set U. Denote by $(W(U), \Omega)$ the smallest $\mathcal{M}$-closed subalgebra of $(V(U), \Omega)$ containing $P(U)$. The algebra $(W(U), \Omega)$ is an $\mathcal{M}$-polyquasigroup. Now assume that (A, Ω) is an $\mathcal{M}$-cancellative strongly entropic algebra. Then obviously there is a surjective homomorphism

$$\pi_{(A,\Omega)} \colon (P(A - \Omega_0), \Omega) \to (A, \Omega)$$

mapping each element from A to itself. Here the set Ω_0 is treated as a subset of A. The kernel of $\pi_{(A,\Omega)}$ is an $\mathcal{M}$-cancellative congruence. Thus, by Lemma 7.4, it may by extended to an $\mathcal{M}$-cancellative congruence $\theta_{(A,\Omega)}$ on $(W(A-\Omega_0),\Omega)$. Now we may define the reflection of (A,Ω) in $\mathcal{M}$-$\boldsymbol{PSE}$ by

$$R^{\mathcal{M}}(A,\Omega) = (W(A-\Omega_0),\Omega)/\theta_{(A,\Omega)}$$

and injective reflecting homomorphism by

$$r^{\mathcal{M}}_{(A,\Omega)} \colon a \mapsto a/\theta_{(A,\Omega)}.$$

Statements (1)-(4) follow from Propositions 7.7-7.10. □

8. Quasi-affine representations

The results obtained up to now allow us to formulate representation theorems, even for non-entropic algebras.

Let $\sigma\colon \Phi \to \mathbb{N}$ be a subtype of τ. We say that (A,Ω) is *σ-entropic* if it satisfies all identities $\varepsilon_{\mu,\omega}$, where $\mu \in \Phi$, $\omega \in \Omega$. We say that (A,Ω) is *σ-strongly entropic* if it is σ-entropic, and satisfies all identities $t_1 \overset{se}{\approx} t_2$, where t_1 and t_2 are σ-terms. Let $\mathcal{N}$ be a monoid of σ-terms. Then

$$\langle \mathcal{N} \rangle = \{t(v,s_1,\dots,s_n) \mid t(v,x_1,\dots,x_n) \in \mathcal{N},\ s_1,\dots,s_n \in \mathrm{Term}(X) - \mathcal{S}\}$$

is the smallest monoid of τ-terms containing $\mathcal{N}$. We say that a monoid $\mathcal{M}$ of τ-terms is *σ-generated (σ-proper)* provided that there exists a (proper) monoid $\mathcal{N}$ of σ-terms such that $\mathcal{M} = \langle \mathcal{N} \rangle$. Note that in such a situation we always have

$$T^{\mathcal{M}}_B(A,\Omega) = T^{\mathcal{N}}_B(A,\Phi).$$

Lemma 8.1. *Let $\mathcal{M}$ be a σ-generated monoid of terms. Then the statements of Lemmas and Propositions 7.2-7.5 hold with entropicity replaced by σ-entropicity.*

Proposition 8.2. *Let $\mathcal{M}$ be a σ-generated monoid of terms. Then the statements of Theorem 7.1 hold when strong entropicity is replaced by σ-strong entropicity.*

Proof. Let $\mathcal{N}$ be monoid of σ-terms such that $\mathcal{M} = \langle \mathcal{N} \rangle$. Let (A,Ω) be $\mathcal{M}$-cancellative σ-strongly entropic algebra. Its reduct (A,Φ) is strongly entropic and $\mathcal{N}$-cancellative. By Theorem 7.1, (A,Φ) embeds into a strongly entropic $\mathcal{N}$-polyquasigroup (B,Φ). We extend the structure of (B,Φ). Let $\omega \in \Omega$. Because (A,Ω) is σ-entropic the composition

$$(A,\Phi)^{\tau(\omega)} \xrightarrow{\omega} (A,\Phi) \overset{\iota}{\hookrightarrow} (B,\Phi)$$

is a homomorphism. We may assume that (A,Φ) is $\mathcal{N}$-dense in (B,Φ). Then $(A^{\tau(\omega)},\Phi)$ is $\mathcal{N}$-dense in $(B^{\tau(\omega)},\Phi)$ and by Proposition 7.5 the mapping $\iota \circ \omega$ may be uniquely extended to the homomorphism $\omega'\colon (B,\Phi)^{\tau(\omega)} \to (B,\Phi)$. Let $\Omega' = \{\omega' \mid \omega \in \Omega\}$. Note that if $\nu \in \Phi$ then, by the uniqueness, ν' coincides with operation ν in (B,Φ). Thus (A,Ω) embeds into (B,Ω'). Moreover, because each ω' is a σ-homomorphism the algebra (B,Ω') is σ-entropic. Finally, since (B,Φ) is an $\mathcal{N}$-polyquasigroup, the algebra (B,Ω') is an $\mathcal{M}$-polyquasigroup. The rest follows from Lemma 8.1. □

$\mathcal{M}$-polyquasigroups may be considered as algebras of an extended type. Namely, let

$$\Omega_{\mathcal{M}} = \{\omega_t \mid t \in \mathcal{M}\} \cup \Omega,$$

and let $\tau_{\mathcal{M}} \colon \Omega_{\mathcal{M}} \to \mathbb{N}$ be the extension of τ given by $\tau_{\mathcal{M}}(\omega_t) = |\arg(t)|$ for $t \in \mathcal{M}$. For an $\mathcal{M}$-polyquasigroup (B, Ω) and a term $t(v, x_1, \dots, x_n) \in \mathcal{M}$, define

$$\omega_t(b, b_1, \dots, b_n) = t(\_, b_1, \dots, b_n)^{-1}(b).$$

Lemma 8.3. *Let $\mathcal{N}$ be a monoid of σ-terms. If an $\langle\mathcal{N}\rangle$-polyquasigroup (B, Ω) is σ-entropic (idempotent), then $(B, \Omega_{\mathcal{N}})$ is $\sigma_{\mathcal{N}}$-entropic (idempotent).*

Proof. Assume that (B, Ω) is σ-entropic. Let t be a (τ-)term and $s \in \mathcal{N}$. Then for all $a_i, b_j^k \in B$

$$\begin{aligned} s(t(a_1, \dots, a_m), t(b_1^1, \dots, b_m^1), \dots, t(b_1^n, \dots, b_m^n)) \\ = t(s(a_1, b_1^1, \dots, b_1^n), \dots, s(a_m, b_m^1, \dots, b_m^n)). \end{aligned}$$

Substituting $c_i = s(a_i, b_i^1, \dots, b_i^n)$ for $i \in \{1, \dots, m\}$ we obtain

$$\begin{aligned} \omega_s(t(c_1, \dots, c_m), t(b_1^1, \dots, b_m^1), \dots, t(b_1^n, \dots, b_m^n)) = t(a_1, \dots, a_m) \\ = t(\omega_s(c_1, b_1^1, \dots, b_1^n), \dots, \omega_s(c_m, b_m^1, \dots, b_m^n)). \end{aligned}$$

So $(B, \Omega_{\mathcal{N}})$ is σ-entropic. Now if $t \in \mathcal{N}$ then similarly we get that $(B, \Omega_{\mathcal{N}})$ satisfy $\varepsilon_{\omega_s, \omega_t}$. Thus, $(B, \Omega_{\mathcal{N}})$ is $\sigma_{\mathcal{N}}$-entropic. Second statement is much easer. □

Recall from [5] that an algebra is *affine (over a ring $(R, +, -, 0, \cdot, 1)$)* if it is polynomially equivalent to a module (over the ring $(R, +, -, 0, \cdot, 1)$). An algebra is *quasi-affine (over a ring $(R, +, -, 0, \cdot, 1)$)* if it is a subreduct of an affine algebra (over the ring $(R, +, -, 0, \cdot, 1)$), see [20, 12]. Let $(\mathbb{Z}\langle\Sigma\rangle, +, -, 0, \cdot, 1)$ be a free noncommutative ring over Σ. Note that a τ-algebra is (quasi-)affine iff it is (quasi-)affine over $(\mathbb{Z}\langle\Sigma\rangle, +, -, 0, \cdot, 1)$. We will use the following not difficult fact (see Theorem 418 in [25] or Theorem 6.2.5 and Corollary 6.3.2 in [23]).

Proposition 8.4. *If a nonempty algebra (A, Ω) has a Mal'cev term operation M which is a homomorphism from (A^3, Ω) into (A, Ω), then it is affine. If moreover (A, Ω) is idempotent, then it is equivalent to an idempotent reduct of a module.*

Lemma 8.5. *An entropic affine algebra is affine over a commutative ring.*

Proof. Let (A, Ω) be an entropic algebra polynomially equivalent to a module $(A, +, -, 0, \mathbb{Z}\langle\Sigma\rangle)$. Consider $(\omega, i), (\nu, j) \in \Sigma$. Let c_ω and c_ν be elements from A such that

$$\begin{aligned} \omega(a_1, \dots, a_{\tau(\omega)}) &= (\omega, 1)a_1 + \dots + (\omega, \tau(\omega))a_{\tau(\omega)} + c_\omega \quad \text{and} \\ \nu(b_1, \dots, b_{\tau(\nu)}) &= (\nu, 1)b_1 + \dots + (\nu, \tau(\nu))b_{\tau(\nu)} + c_\nu \end{aligned}$$

for all $a_1, \dots, a_{\tau(\omega)}, b_1, \dots, b_{\tau(\nu)} \in A$. We have

$$\begin{aligned} (\omega, 1)c_\nu + \dots + (\omega, \tau(\omega))c_\nu + c_\omega &= \omega(\nu(0, \dots, 0), \dots, \nu(0, \dots, 0)) \\ &= \nu(\omega(0, \dots, 0), \dots, \omega(0, \dots, 0)) \\ &= (\nu, 1)c_\omega + \dots + (\nu, \tau(\nu))c_\omega + c_\nu \end{aligned}$$

and for $a \in A$

$$\begin{aligned}(\omega,i)(\nu,j)a &+ (\omega,1)c_\nu + \cdots + (\omega,\tau(\omega))c_\nu + c_\omega \\ &= \omega(\nu(0,\ldots,0),\ldots,\nu(0,\ldots,a,\ldots,0)\ldots\ldots,\nu(0,\ldots,0)) \\ &= \nu(\omega(0,\ldots,0),\ldots,\omega(0,\ldots,a,\ldots,0)\ldots\ldots,\omega(0,\ldots,0)) \\ &= (\nu,j)(\omega,i)a + (\nu,1)c_\omega + \cdots + (\nu,\tau(\nu))c_\omega + c_\nu.\end{aligned}$$

Thus $(\omega,i)(\nu,j)a = (\nu,j)(\omega,i)a$ for all $a \in A$ and (A,Ω) is polynomially equivalent to $(A,+,-,0,\mathbb{Z}[\Sigma])$. □

Theorem 8.6. *Let (A,Ω) be an algebra. Let $\eta(x,y,z)$ be a term such that (A,Ω) satisfies the quasi-identities*

$$\eta(x_1,y,z) \approx \eta(x_2,y,z) \rightarrow x_1 \approx x_2 \quad \text{and} \quad \eta(x,y_1,z) \approx \eta(x,y_2,z) \rightarrow y_1 \approx y_2.$$

Then

1. *if (A,Ω) satisfies the identities $\varepsilon_{\eta,\omega}$, $\omega \in \Omega$, then it is quasi-affine;*
2. *if (A,Ω) is entropic, then it is quasi-affine over a commutative ring;*
3. *if (A,Ω) is idempotent and satisfies the identities $\varepsilon_{\eta,\omega}$, $\omega \in \Omega$, then it is a subreduct of a module;*
4. *if (A,Ω) is a mode, then it is a subreduct of a module over a commutative ring;*

Remark 8.7. Point (1) was proved by K. Kearnes in [12] with the additional assumption that (A,Ω) is abelian. Point (4) is a slight generalization of Romanowska-Smith Theorem 1.1

Proof. Assume that η is a symbol of basic operation, that is $\eta \in \Omega$. Let $\Phi = \{\eta\}$ and $\sigma = \tau|_\Phi$. Let $\mathcal{N}$ be the monoid of σ-terms generated by $\{\eta(v,y,z),\eta(x,v,z)\}$. Then the assumption of Theorem says that (A,Ω) is $\langle\mathcal{N}\rangle$-cancellative.
(1) $\mathcal{N}$ is proper and satisfies (Ax) (as a monoid of σ-terms). Thus, by Theorem 6.1, (A,Ω) is σ-strongly entropic and by Proposition 8.2 it embeds into a σ-entropic $\langle\mathcal{N}\rangle$-polyquasigroup (B,Ω). The algebra $(B,\Omega_\mathcal{N})$ has a Mal'cev $\sigma_\mathcal{N}$-term

$$M(x,y,z) = \eta(\omega_{\eta(v,y,z)}(x,z,z),\omega_{\eta(x,v,z)}(\omega_{\eta(v,y,z)}(y,z,z),z,y),y),$$

so by Lemma 8.3 and Proposition 8.4, $(B,\Omega_\mathcal{N})$ is affine or empty. Thus (A,Ω) is quasi-affine.
(2) If (A,Ω) is entropic, then by Proposition 8.2, (B,Ω) is entropic and by Lemma 8.3, $(B,\Omega_\mathcal{N})$ is entropic. Moreover $(B,\Omega_\mathcal{N})$ is affine, hence by Lemma 8.5, it is affine over a commutative ring or empty. Thus (A,Ω) is quasi-affine over a commutative ring.
Proofs of (3) and (4) are similar. □

9. Full generality

We say that an algebra (A,Ω) is *entropically abelian* if for each pair of terms $t_1(v,x_1,\ldots,x_n)$, $t_2(v,y_1,\ldots,y_m)$, where $\bar{a}(t_1,v) = \bar{a}(t_2,v)$, the quasi-identity

$$t_1(v,x_1,\ldots,x_n) \approx t_2(v,y_1\ldots,y_m) \rightarrow t_1(u,x_1,\ldots,x_n) \approx t_2(u,y_1\ldots,y_m) \quad \text{(ETC)}$$

holds in (A, Ω).

Lemma 9.1. *If (A, Ω) is quasi-affine over a commutative ring then it is entropically abelian.*

Proof. Let (A, Ω) be polynomially equivalent to a module $(A, +, -, 0, \mathbb{Z}[\Sigma])$. Let t_1, t_2 be as in (ETC). Assume that

$$t_1(a, c_1, \dots, c_n) = t_2(a, d_1, \dots, d_m).$$

Let $c, d \in A$ be such that

$$t_1(a, c_1, \dots, c_n) = \bar{a}(t_1, x)a + c$$

and

$$t_2(a, d_1, \dots, d_m) = \bar{a}(t_2, x)a + d.$$

Since $\bar{a}(t_1, x) = \bar{a}(t_2, x)$, we have $c = d$. Consequently

$$t_1(b, c_1, \dots, c_n) = \bar{a}(t_1, x)b + c = \bar{a}(t_2, x)b + d = t_2(b, d_1, \dots, d_m).$$

Now Lemma follows from the fact that a subreduct of an entropically abelian algebra is entropically abelian, too. □

Lemma 9.2. *Let $\mathcal{M}$ be a proper monoid of terms. Then each $\mathcal{M}$-cancellative entropic algebra is entropically abelian.*

Proof. It follows from Theorem 8.6 (2) and Lemma 9.1. □

Theorem 9.3. *Let $\mathcal{M}$ be a proper monoid of terms. Then the class $\mathcal{M}$-$\boldsymbol{CE}$ generates the variety $\boldsymbol{SE}$.*

Proof. The only place where we used the condition (Ax) in the proof of Theorem 6.1 is Lemma 6.7. We will reprove the statement of it without using (Ax). Let $t_1(x_1, \dots, x_n) \overset{se}{\approx} t_2(x_1, \dots, x_n)$. By Lemma 6.6, there exist linear $\tau_{>0}$-terms s_i, with mutually disjoint sets of variables, such that $t_1(s_1, \dots, s_n) \overset{\mathcal{M}}{\approx} t_2(s_1, \dots, s_n)$. Note that $\bar{a}(t_1, x_i) = \bar{a}(t_1, x_i)$ and by Lemma 9.2 we may apply (ETC). Thus

$$t_1(x_1, \dots, x_n) \overset{\mathcal{M}}{\approx} t_2(x_1, \dots, x_n).$$

□

Remark 9.4. Note that with the aid of (ETC), Corollary 6.4 follows from Lemma 6.2 immediately. Thus, assuming entropic abelianness we may remove the usage of (P) from the proof of Theorem 6.1 as well as the usage of (Ax). Thus we get that the class of entropically abelian entropic algebras generates $\boldsymbol{SE}$, a result very similar to Theorem 3.3 in [17]. In the case when $|\Omega_{>1}| = 1$, with small changes in the proof, we get that the class of abelian entropic algebras generates $\boldsymbol{SE}$. This is Theorem 2.9 in [17].

Finally, we can formulate our embedability result in full strength. Let us denote the class of all $\mathcal{M}$-cancellative σ-entropic algebras by $\mathcal{M}$-$\boldsymbol{CE_\sigma}$, and the class of all σ-entropic $\mathcal{M}$-polyquasigroups by $\mathcal{M}$-$\boldsymbol{PE_\sigma}$.

Theorem 9.5. *For a σ-proper monoid $\mathcal{M}$ of terms, there exists an injective reflector from the category of $\mathcal{M}$-cancellative σ-entropic algebras into the category of σ-entropic $\mathcal{M}$-polyquasigroups*

$$R^{\mathcal{M}}\colon \mathcal{M}\text{-}\boldsymbol{CE_\sigma} \to \mathcal{M}\text{-}\boldsymbol{PE_\sigma}.$$

In particular, a σ-entropic $\mathcal{M}$-cancellative algebra embeds into a σ-entropic $\mathcal{M}$-polyquasigroup. Moreover, for $(A, \Omega) \in \mathcal{M}$-$\boldsymbol{CSE}$:

(1) *(A, Ω) satisfies precisely the same identities as $R^{\mathcal{M}}(A, \Omega)$;*
(2) *if (A, Ω) is subdirectly irreducible then $R^{\mathcal{M}}(A, \Omega)$ is subdirectly irreducible;*

and if additionally (A, Ω) is idempotent then

(3) *(A, Ω) and $R^{\mathcal{M}}(A, \Omega)$ satisfy precisely the same first order universal sentences;*
(4) *if (A, Ω) is simple, then $R^{\mathcal{M}}(A, \Omega)$ is simple.*

Proof. By Theorem 9.3 each σ-entropic $\mathcal{M}$-cancellative algebra is σ-strongly entropic. Thus, we may apply Proposition 8.2. □

10. Final remarks

Recall from the literature that each distributive (paramedial, trimedial) cancellative groupoid may be embedded into a distributive (paramedial, trimedial) quasigroup (see [13], [11] and [13]). A groupoid is *paramedial* if it satisfies the identity $(xy)(zt) \approx (ty)(zx)$. A groupoid is *trimedial* if each of its three generated subgroupoid is entropic (in our sense). Recall also from general algebra the localization of modules over arbitrary rings. These cases are not in the scope of Theorem 9.5. Summarizing, we have just touched the problem of embedding $\mathcal{M}$-cancellative algebras into $\mathcal{M}$-polyquasigroups.

Problem 10.1. When does an $\mathcal{M}$-cancellative algebra in a variety $\boldsymbol{V}$ embed into an $\mathcal{M}$-polyquasigroup in $\boldsymbol{V}$?

Next problem is technical.

Problem 10.2. Find a syntactic proof that cancellative entropic algebras are (entropically) abelian.

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Faculty of Mathematics and Information Sciences, Warsaw University of Technology, 00-661 Warsaw, Poland
E-mail address: m.stronkowski@mini.pw.edu.pl
URL: www.mini.pw.edu.pl/~stronkow